

A NEW CLASS OF LAGUERRE POLY-BERNOULLI NUMBERS AND POLYNOMIALS

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ABSTRACT. In this paper, we introduce a new class of Laguerre poly-Bernoulli polynomials and give some identities of these polynomials related to the Stirling numbers of the second kind. We derive some implicit summation formulae and general symmetry identities are derived by using different analytical means and applying generating functions.

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1. INTRODUCTION

The two variable Laguerre polynomials $L_n(x, y)$ are defined by the generating function [5]

$$(1) \quad e^{yt} C_0(xt) = \sum_{n=0}^{\infty} L_n(x, y) \frac{t^n}{n!},$$

where $C_0(x)$ is the 0-th order Tricomi function [14]

$$(2) \quad C_0(x) = \sum_{r=0}^{\infty} \frac{(-1)^r x^r}{(r!)^2}$$

and are represented by the series

$$(3) \quad L_n(x, y) = \sum_{s=0}^n \frac{n!(-1)^s y^{n-s} x^s}{(n-s)!(s!)^2}$$

As it is well known, the Bernoulli polynomials are defined by their generating function

$$(4) \quad \left(\frac{t}{e^t - 1} \right) e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad (\text{see [1] – [15]})$$

When $x = 0$, $B_n = B_n(0)$ are called the Bernoulli numbers. From (4), we have

$$(5) \quad B_n(x) = \sum_{m=0}^n \binom{n}{m} B_{n-m} x^m$$

The classical polylogarithm function $Li_k(z)$ is

$$(6) \quad Li_k = \sum_{m=1}^{\infty} \frac{z^m}{m!}, \quad |z| < 1 \quad (\text{see [6], [7]})$$

so for $k \leq 1$,

$$Li_k = -\ln(1-z), \quad Li_0(z) = \frac{z}{1-z}, \quad Li_{-1} = \frac{z}{(1-z)^2}, \quad \dots$$

The poly-Bernoulli polynomials are given by

$$(7) \quad \frac{Li_k(1-e^{-t})}{e^t-1} e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{t^n}{n!}, \quad (\text{see [6], [8] - [11]})$$

For $k = 1$, we have

$$(8) \quad \frac{Li_1(1-e^{-t})}{e^t-1} e^{xt} = \frac{t}{e^t-1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!},$$

From (4) and (8), we obtain

$$B_n^{(1)}(x) = B_n(x), \quad (n \geq 0)$$

Very recently, Pathan et al. [12] introduced the generalized Hermite-Bernoulli polynomials of two variables ${}_H B_n^{(\alpha)}(x, y)$ is defined by

$$(9) \quad \left(\frac{t}{e^t-1} \right)^{\alpha} e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_H B_n^{(\alpha)}(x, y) \frac{t^n}{n!}$$

which are essentially generalization of Bernoulli numbers, Bernoulli polynomials, Hermite polynomials and Hermite-Bernoulli polynomials ${}_H B_n(x, y)$ introduced by Dattoli et al. ([4], p. 386 (1.6)) in the form

$$(10) \quad \left(\frac{t}{e^t-1} \right) e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_H B_n(x, y) \frac{t^n}{n!}$$

The stirling number of the first kind is given by

$$(11) \quad (x)_n = x(x-1) \cdots (x-n+1) = \sum_{l=0}^n S_1(n, l) x^l, \quad (n \geq 0)$$

and the stirling number of the second kind is defined by generating function to be

$$(12) \quad (e^t-1)^n = n! \sum_{l=n}^{\infty} S_2(l, n) \frac{t^l}{l!}$$

The 2-variable Kampé de Fériet generalization of the Hermite polynomials [3] and [4] reads

$$(13) \quad H_n(x, y) = n! \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \frac{y^r x^{n-2r}}{r!(n-2r)!}$$

These polynomials are usually defined by the generating function

$$(14) \quad e^{xt+yt^2} = \sum_{n=0}^{\infty} H_n(x, y) \frac{t^n}{n!}$$

and reduce to the ordinary Hermite polynomials $H_n(x)$ (see [1]) when $y = -1$ and x is replaced by $2x$.

In this paper, first we give the definition of the Laguerre poly-Bernoulli polynomials ${}_LB_n^{(k)}(x, y, z)$ and we have given some formulae of those polynomials related to the Stirling numbers of second kind. Some implicit summation formulae and general symmetry identities are derived by using different analytical means and applying generating functions. These result extended some known summations and identities of generalized Hermite-Bernoulli polynomials studied by Dattoli et al., Khan, and Pathan and Khan.

2. A NEW CLASS OF LAGUERRE POLY-BERNOULLI NUMBERS AND POLYNOMIALS

Now, we define the Laguerre poly-Bernoulli polynomials as follows:

$$(15) \quad \frac{Li_k(1 - e^{-t})}{e^t - 1} e^{yt+zt^2} C_0(xt) = \sum_{n=0}^{\infty} {}_LB_n^{(k)}(x, y, z) \frac{t^n}{n!}, \quad (k \in \mathbb{Z})$$

so that

$$(16) \quad {}_LB_n^{(k)}(x, y, z) = \sum_{m=0}^n \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{B_{n-m}^{(k)} L_{m-2k}(x, y) z^k n!}{(m-2k)! k! (n-m)!}$$

when $x = y = z = 0$, $B_n^{(k)} = {}_LB_n^{(k)}(0, 0, 0)$ are called the poly-Bernoulli numbers. By (15), we easily get $B_n^{(k)} = 0$. For $k = 1$, from (15), we have

$$(17) \quad \frac{Li_1(1 - e^{-t})}{e^t - 1} e^{yt+zt^2} C_0(xt) = \sum_{n=0}^{\infty} {}_LB_n(x, y, z) \frac{t^n}{n!}, \quad (k \in \mathbb{Z})$$

Thus by (15) and (17), we get

$${}_LB_n^{(k)}(x, y, z) = {}_LB_n(x, y, z), \quad (n \geq 0).$$

Theorem 2.1. For $n \geq 0$, we have

$$(18) \quad {}_LB_n^{(2)}(x, y, z) = \sum_{m=0}^n \binom{n}{m} \frac{B_m}{m+1} {}_LB_{n-m}(x, y, z)$$

Proof. Applying definition (15), we have

$$\begin{aligned} & \frac{Li_k(1 - e^{-t})}{e^t - 1} e^{yt+zt^2} C_0(xt) = \sum_{n=0}^{\infty} {}_LB_n^{(k)}(x, y, z) \frac{t^n}{n!} \\ & = \left(\frac{1}{e^t - 1} \right) e^{yt+zt^2} C_0(xt) \int_0^t \frac{1}{e^z - 1} \int_0^t \frac{1}{e^z - 1} \cdots \frac{1}{e^z - 1} \int_0^t \frac{z}{e^z - 1} dz dz \cdots dz \end{aligned}$$

In particular $k = 2$, we have

$$\begin{aligned} {}_L B_n^{(2)}(x, y, z) &= \left(\frac{1}{e^t - 1} \right) e^{yt+zt^2} C_0(xt) \int_0^t \frac{z}{e^z - 1} \\ &= \left(\sum_{m=0}^{\infty} \frac{B_m t^m}{m+1} \right) \left(\frac{t}{e^t - 1} \right) e^{yt+zt^2} C_0(xt) \\ &= \left(\sum_{m=0}^{\infty} \frac{B_m t^m}{m+1} \right) \left(\sum_{n=0}^{\infty} {}_L B_n(x, y, z) \frac{t^n}{n!} \right) \end{aligned}$$

Replacing n by $n - m$ in the above equation, we have

$$= \sum_{n=0}^{\infty} \sum_{m=0}^n \binom{n}{m} \frac{B_m}{m+1} {}_L B_{n-m}(x, y, z) \frac{t^n}{n!}$$

On equating the coefficients of the like power of t in the above equation, we get the result (18).

Theorem 2.2 For $n \geq 1$, we have

$$(19) \quad {}_L B_n^{(k)}(x, y, z) = \sum_{m=0}^n \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{B_m^{(k)} L_{n-m-2k}(x, y) z^k n!}{m! k! (n-m-2k)!}$$

Proof. From equation (15), we have

$$\begin{aligned} \sum_{n=0}^{\infty} {}_L B_n^{(k)}(x, y, z) \frac{t^n}{n!} &= \frac{Li_k(1 - e^{-t})}{e^t - 1} e^{yt+zt^2} C_0(xt) \\ &= \left(\sum_{m=0}^{\infty} B_m^{(k)} \frac{t^m}{m!} \right) \left(\sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{L_{n-2k}(x, y) z^k}{k! (n-2k)!} t^n \right) \end{aligned}$$

Replacing n by $n - m$ in the above equation and comparing the coefficients of t^n , we get the result (19).

Theorem 2.3 For $n \geq 0$, we have

$$(20) \quad {}_L B_n^{(k)}(x, y, z) = \sum_{p=0}^n \sum_{l=1}^{p+1} \frac{(-1)^{l+p+1} l! S_2(p+1, l)}{l^k (p+1)} \binom{n}{p} {}_L B_{n-p}(x, y, z)$$

Proof. From equation (15), we have

$$(21) \quad \sum_{n=0}^{\infty} {}_L B_n^{(k)}(x, y, z) \frac{t^n}{n!} = \left(\frac{Li_k(1 - e^{-t})}{t} \right) \left[\left(\frac{t}{e^t - 1} \right) e^{yt+zt^2} C_0(xt) \right]$$

Now

$$\begin{aligned} \frac{1}{t} Li_k(1 - e^{-t}) &= \frac{1}{t} \sum_{l=1}^{\infty} \frac{(1 - e^{-t})^l}{l^k} = \frac{1}{t} \sum_{l=1}^{\infty} \frac{(-1)^l}{l^k} (1 - e^{-t})^l \\ &= \frac{1}{t} \sum_{l=1}^{\infty} \frac{(-1)^l}{l^k} l! \sum_{p=l}^{\infty} (-1)^p S_2(p, l) \frac{t^p}{p!} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{t} \sum_{p=1}^{\infty} \sum_{l=l}^p \frac{(-1)^{l+p}}{l^k} l! S_2(p, l) \frac{t^p}{p!} \\
(22) \quad &= \sum_{p=0}^{\infty} \left(\sum_{l=l}^{p+1} \frac{(-1)^{l+p+1}}{l^k} l! \frac{S_2(p+1, l)}{p+1} \right) \frac{t^p}{p!}
\end{aligned}$$

From equation (21) and (22), we get

$$\sum_{n=0}^{\infty} {}_L B_n^{(k)}(x, y, z) \frac{t^n}{n!} = \sum_{p=0}^{\infty} \left(\sum_{l=l}^{p+1} \frac{(-1)^{l+p+1}}{l^k} l! \frac{S_2(p+1, l)}{p+1} \right) \frac{t^p}{p!} \left(\sum_{n=0}^{\infty} {}_L B_n(x, y, z) \frac{t^n}{n!} \right)$$

Replacing n by $n - p$ in the r.h.s of above equation and comparing the coefficients of t^n , we get the result (20).

Theorem 2.4 For $n \geq 1$, we have

$$(23) \quad {}_L B_n^{(k)}(x, y+1, z) - {}_L B_n^{(k)}(x, y, z) = \sum_{p=1}^n \sum_{l=1}^p \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^{l+p}}{l^k} l! n! S_2(p, l) \frac{{}_L B_{n-p-2k}(x, y) z^k}{p! k! (n-p-2k)!}$$

Proof. Using the definition (15), we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} {}_L B_n^{(k)}(x, y+1, z) \frac{t^n}{n!} - \sum_{n=0}^{\infty} {}_L B_n^{(k)}(x, y, z) \frac{t^n}{n!} \\
&= \frac{Li_k(1 - e^{-t})}{e^t - 1} e^{(y+1)t+zt^2} C_0(xt) - \frac{Li_k(1 - e^{-t})}{e^t - 1} e^{yt+zt^2} C_0(xt) \\
&= Li_k(1 - e^{-t}) e^{yt+zt^2} C_0(xt) \\
&= \sum_{p=1}^{\infty} \left(\sum_{l=1}^p \frac{(-1)^{l+p}}{l^k} l! S_2(p, l) \right) \frac{t^p}{p!} e^{yt+zt^2} C_0(xt) \\
&= \sum_{p=1}^{\infty} \left(\sum_{l=1}^p \frac{(-1)^{l+p}}{l^k} l! S_2(p, l) \right) \frac{t^p}{p!} \left(\sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{{}_L B_{n-2k}(x, y) z^k}{k! (n-2k)!} t^n \right)
\end{aligned}$$

Replacing n by $n - p$ in the above equation and comparing the coefficients of t^n , we get the result (23).

Theorem 2.5 For $d \in N$ with $d \equiv 1(mod 2)$, we have

$$(24) \quad {}_L B_n^{(k)}(x, y, z) = \sum_{p=0}^n \binom{n}{p} d^{n-p-1} \sum_{l=0}^{p+1} \sum_{a=0}^{d-1} \frac{(-1)^{l+p+1} l! S_2(p+1, l)}{l^k} (-1)^a {}_L B_{n-p} \left(x, \frac{a+y}{d}, z \right)$$

Proof. From equation (15), we have

$$\sum_{n=0}^{\infty} {}_L B_n^{(k)}(x, y, z) \frac{t^n}{n!} = \frac{Li_k(1 - e^{-t})}{e^t - 1} e^{yt+zt^2} C_0(xt)$$

$$\begin{aligned}
&= \left(\frac{Li_k(1 - e^{-t})}{t} \right) \left(\frac{t}{e^{dt} - 1} \sum_{a=0}^{d-1} e^{(a+y)t+zt^2} C_0(xt) \right) \\
&= \left(\sum_{p=0}^{\infty} \left(\sum_{l=1}^{p+1} \frac{(-1)^{l+p+1}}{l^k} l! \frac{S_2(p+1, l)}{p+1} \right) \frac{t^p}{p!} \right) \left(\sum_{m=0}^{\infty} d^{m-1} \sum_{a=0}^{d-1} (-1)^a {}_L B_n \left(x, \frac{a+y}{d}, z \right) \frac{t^n}{n!} \right)
\end{aligned}$$

Replacing n by $n - p$ in the above equation and comparing the coefficients of t^n , we get the result (24).

3. IMPLICIT SUMMATION FORMULAE INVOLVING LAGUERRE POLY-BERNOULLI POLYNOMIALS

This section of the paper is devoted to employing the definition of the Laguerre poly-Bernoulli polynomials ${}_L B_n^{(k)}(x, y, z)$ to obtain finite summations. For the derivation of implicit formulae involving the Laguerre poly-Bernoulli polynomials ${}_L B_n^{(k)}(x, y, z)$ the same consideration as developed for the ordinary Hermite and related polynomials in Khan [7] and Pathan [12]-[13] holds as well. First we prove the following results involving Laguerre poly-Bernoulli polynomials ${}_L B_n^{(k)}(x, y, z)$.

Theorem 3.1. For $x, y, z \in R$ and $n \geq 0$, The following implicit summation formula for Laguerre poly-Bernoulli polynomials ${}_L B_n^{(k)}(x, y, z)$ holds true:

$$(25) \quad {}_L B_{l+p}^{(k)}(x, v, z) = \sum_{m,n=0}^{l,p} \binom{l}{m} \binom{p}{n} (v-y)^{m+n} {}_L B_{l+p-m-n}^{(k)}(x, y, z)$$

Proof. We replace t by $t + u$ and rewrite the generating function (15) as

$$(26) \quad \left(\frac{Li_k(1 - e^{-(t+u)})}{e^{t+u} - 1} \right) e^{z(t+u)^2} C_0(x(t+u)) = e^{-y(t+u)} \sum_{l,p=0}^{\infty} {}_L B_{l+p}^{(k)}(x, y, z) \frac{t^l}{l!} \frac{u^p}{p!}$$

Replacing y by v in the above equation and equating the resulting equation to the above equation, we get

$$(27) \quad e^{(v-y)(t+u)} \sum_{l,p=0}^{\infty} {}_L B_{l+p}^{(k)}(x, y, z) \frac{t^l}{l!} \frac{u^p}{p!} = \sum_{l,p=0}^{\infty} {}_L B_{l+p}^{(k)}(x, v, z) \frac{t^l}{l!} \frac{u^p}{p!}$$

on expanding exponential function (27) gives

$$(28) \quad \sum_{N=0}^{\infty} \frac{[(v-y)(t+u)]^N}{N!} \sum_{l,p=0}^{\infty} {}_L B_{l+p}^{(k)}(x, y, z) \frac{t^l}{l!} \frac{u^p}{p!} = \sum_{l,p=0}^{\infty} {}_L B_{l+p}^{(k)}(x, v, z) \frac{t^l}{l!} \frac{u^p}{p!}$$

which on using formula [[15], p. 52(2)]

$$(29) \quad \sum_{N=0}^{\infty} f(N) \frac{(x+y)^N}{N!} = \sum_{m,n=0}^{\infty} f(m+n) \frac{x^m}{m!} \frac{y^n}{n!}$$

in the left hand side becomes

$$(30) \quad \sum_{m,n=0}^{\infty} \frac{(v-y)^{m+n} t^m u^n}{m!n!} \sum_{l,p=0}^{\infty} {}_L B_{l+p}^{(k)}(x, y, z) \frac{t^l}{l!} \frac{u^p}{p!} = \sum_{l,p=0}^{\infty} {}_L B_{l+p}^{(k)}(x, v, z) \frac{t^l}{l!} \frac{u^p}{p!}$$

Now replacing l by $l-m$, p by $p-n$ and using the lemma [[15], p. 100(1)] in the left hand side of (30), we get

$$(31) \quad \sum_{m,n=0}^{\infty} \sum_{l,p=0}^{\infty} \frac{(v-y)^{m+n}}{m!n!} {}_L B_{l+p-m-n}^{(k)}(x, y, z) \frac{t^l}{(l-m)!} \frac{u^p}{(p-n)!} = {}_L B_{l+p}^{(k)}(x, v, z) \frac{t^l}{l!} \frac{u^p}{p!}$$

Finally, on equating the coefficients of the like powers of t and u in the above equation, we get the required result.

Remark 1. By taking $l = 0$ in equation (25), we immediately deduce the following corollary.

Corollary 3.1. The following implicit summation formula for Laguerre poly-Bernoulli polynomials ${}_L B_n^{(k)}(x, v, z)$ holds true:

$$(32) \quad {}_L B_p^{(k)}(x, v, z) = \sum_{n=0}^p \binom{p}{n} (v-y)^n {}_L B_{p-n}^{(k)}(x, y, z)$$

Remark 2. On replacing v by $v+y$ and setting $x = z = 0$ in Theorem (3.1), we get the following result involving Laguerre poly-Bernoulli polynomial of one variable

$$(33) \quad {}_L B_{l+p}^{(k)}(v+y) = \sum_{m,n=0}^{l,p} \binom{l}{m} \binom{p}{n} (v)^{m+n} {}_L B_{l+p-m-n}^{(k)}(y)$$

whereas by setting $v = 0$ in Theorem (3.1), we get another result involving Laguerre poly-Bernoulli polynomial of one and two variable

$$(34) \quad {}_L B_{l+p}^{(k)}(x, z) = \sum_{m,n=0}^{l,p} \binom{l}{m} \binom{p}{n} (-y)^{m+n} {}_L B_{l+p-m-n}^{(k)}(x, y, z)$$

Remark 3. Along with the above result we will exploit extended forms of Laguerre poly-Bernoulli polynomial ${}_L B_{l+p}^{(k)}(x, v)$ by setting $z = 0$ in the Theorem (3.1) to get

$$(35) \quad {}_L B_{l+p}^{(k)}(x, v) = \sum_{m,n=0}^{l,p} \binom{l}{m} \binom{p}{n} (v-y)^n {}_L B_{l+p-m-n}^{(k)}(x, y)$$

Theorem 3.2. For $x, y, z \in R$ and $n \geq 0$. Then

$$(36) \quad {}_L B_n^{(k)}(x, y+u, z) = \sum_{j=0}^n \binom{n}{j} u^j {}_L B_{n-j}^{(k)}(x, y, z)$$

Proof. Since

$$\sum_{n=0}^{\infty} {}_L B_n^{(k)}(x, y+u, z) \frac{t^n}{n!} = \frac{Li_k(1-e^{-t})}{e^t-1} e^{(y+u)t+zt^2} C_0(xt)$$

$$\sum_{n=0}^{\infty} {}_L B_n^{(k)}(x, y+u, z) \frac{t^n}{n!} = \left(\sum_{n=0}^{\infty} {}_L B_n^{(k)}(x, y, z) \frac{t^n}{n!} \right) \left(\sum_{j=0}^{\infty} u^j \frac{t^j}{j!} \right)$$

Now, replacing n by $n-j$ and comparing the coefficients of t^n , we get the result (36).

Theorem 3.3. For $x, y, z \in R$ and $n \geq 0$. Then

$$(37) \quad {}_L B_n^{(k)}(x, y+u, z+w) = \sum_{m=0}^n \binom{n}{m} {}_L B_{n-m}^{(k)}(x, y, z) H_m(u, w)$$

Proof. By the definition of Laguerre poly-Bernoulli polynomials and the definition (14), we have

$$\frac{Li_k(1-e^{-t})}{e^t-1} e^{(y+u)t+(z+w)t^2} C_0(xt) = \left(\sum_{n=0}^{\infty} {}_L B_n^{(k)}(x, y, z) \frac{t^n}{n!} \right) \left(\sum_{m=0}^{\infty} H_m(u, w) \frac{t^m}{m!} \right)$$

Now, replacing n by $n-m$ and comparing the coefficients of t^n , we get the result (37).

Theorem 3.4. For $x, y, z \in R$ and $n \geq 0$. Then

$$(38) \quad {}_L B_n^{(k)}(x, y, z) = \sum_{m=0}^{n-2j} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \frac{B_m^{(k)} L_{n-m-2j}(x, y) z^j n!}{m! j! (n-m-2j)!}$$

Proof. Applying the definition (15) to the term $\frac{Li_k(1-e^{-t})}{e^t-1}$ and expanding the exponential and tricorni function $e^{yt+zt^2} C_0(xt)$ at $t=0$ yields

$$\begin{aligned} \frac{Li_k(1-e^{-t})}{e^t-1} e^{yt+zt^2} C_0(xt) &= \left(\sum_{m=0}^{\infty} B_m^{(k)} \frac{t^m}{m!} \right) \left(\sum_{n=0}^{\infty} L_n(x, y) \frac{t^n}{n!} \right) \left(\sum_{j=0}^{\infty} z^j \frac{t^{2j}}{j!} \right) \\ \sum_{n=0}^{\infty} {}_L B_n^{(k)}(x, y, z) \frac{t^n}{n!} &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^n \frac{B_m^{(k)} L_{n-m}(x, y)}{(n-m)! m!} \right) t^n \left(\sum_{j=0}^{\infty} z^j \frac{t^{2j}}{j!} \right) \end{aligned}$$

Now, replacing n by $n-2j$ and comparing the coefficients of t^n , we get the result (38).

Theorem 3.5. For $x, y, z \in R$ and $n \geq 0$. Then

$$(39) \quad {}_L B_n^{(k)}(x, y+1, z) = \sum_{m,j=0}^n \frac{n! (-1)^j (x)^j {}_H B_{n-m-j}^{(k)}(y, z)}{(n-m-j)! m! (j!)^2}$$

Proof. By the definition of Laguerre poly-Bernoulli polynomials, we have

$$\begin{aligned} \sum_{n=0}^{\infty} {}_L B_n^{(k)}(x, y+1, z) \frac{t^n}{n!} &= \frac{Li_k(1-e^{-t})}{e^t-1} e^{(y+1)t+zt^2} C_0(xt) \\ &= \left(\sum_{n=0}^{\infty} \left(\sum_{m=0}^n \frac{{}_H B_{n-m}^{(k)}(y, z)}{(n-m)! n!} \right) t^n \right) \left(\sum_{j=0}^{\infty} \frac{(-1)^j (xt)^j}{(j!)^2} \right) \end{aligned}$$

$$= \left(\sum_{n=0}^{\infty} \left(\sum_{j=0}^{\infty} \sum_{m=0}^n \frac{(-1)^j (x)^j {}_H B_{n-m}^{(k)}(y, z)}{(n-m)! n! (j!)^2} \right) t^{n+j} \right)$$

Replacing n by $n-j$, we have

$$\sum_{n=0}^{\infty} {}_L B_n^{(k)}(x, y+1, z) \frac{t^n}{n!} = \left(\sum_{n=0}^{\infty} \left(\sum_{m,j=0}^n \frac{(-1)^j (x)^j {}_H B_{n-m}^{(k)}(y, z)}{(n-m)! n! (j!)^2} \right) t^{n+j} \right)$$

on comparing the coefficients of t^n , we get the result (39).

Theorem 3.6. The following implicit summation formula for Laguerre poly-Bernoulli polynomials ${}_L B_n^{(k)}(x, y, z)$ holds true:

$$(40) \quad {}_L B_n^{(k)}(x, y+1, z) = \sum_{m=0}^n \binom{n}{m} {}_L B_{n-m}^{(k)}(x, y, z)$$

Proof. By the definition of Laguerre poly-Bernoulli polynomials, we have

$$\begin{aligned} & \sum_{n=0}^{\infty} {}_L B_n^{(k)}(x, y+1, z) \frac{t^n}{n!} + \sum_{n=0}^{\infty} {}_L B_n^{(k)}(x, y, z) \frac{t^n}{n!} \\ &= \left(\frac{Li_k(1-e^{-t})}{e^t-1} \right) (e^t+1) e^{yt+zt^2} C_0(xt) \\ &= \sum_{n=0}^{\infty} {}_L B_n^{(k)}(x, y, z) \frac{t^n}{n!} \left(\sum_{m=0}^{\infty} \frac{t^m}{m!} + 1 \right) \\ &= \sum_{n=0}^{\infty} {}_L B_n^{(k)}(x, y, z) \frac{t^n}{n!} \sum_{m=0}^{\infty} \frac{t^m}{m!} + \sum_{n=0}^{\infty} {}_L B_n^{(k)}(x, y, z) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^n {}_L B_{n-m}^{(k)}(x, y, z) \frac{t^n}{m!(n-m)!} + \sum_{n=0}^{\infty} {}_L B_n^{(k)}(x, y, z) \frac{t^n}{n!} \end{aligned}$$

Finally equating the coefficients of the like powers of t^n , we get the result (40).

Theorem 3.7. The following implicit summation formula for Laguerre poly-Bernoulli polynomials ${}_L B_n^{(k)}(x, y, z)$ holds true:

$$(41) \quad {}_L B_n^{(k)}(x, -y, z) = (-1)^n {}_L B_n^{(k)}(x, y, z)$$

Proof. We replace $-t$ by t in (15) and then subtract the result from (15) itself finding

$$\begin{aligned} & e^{zt^2} \left[\left(\frac{Li_k(1-e^{-t})}{e^t-1} \right) e^{yt} C_0(xt) - \left(\frac{Li_k(1-e^{-t})}{e^t-1} \right) e^{-yt} C_0(-xt) \right] \\ &= \sum_{n=0}^{\infty} [1 - (-1)^n] {}_L B_n^{(k)}(x, y, z) \frac{t^n}{n!} \end{aligned}$$

which is equivalent to

$$\sum_{n=0}^{\infty} {}_L B_n^{(k)}(x, y, z) \frac{t^n}{n!} - \sum_{n=0}^{\infty} {}_L B_n^{(k)}(x, -y, z) \frac{t^n}{n!} = \sum_{n=0}^{\infty} [1 - (-1)^n] {}_L B_n^{(k)}(x, y, z) \frac{t^n}{n!}$$

and thus equating coefficients of the like powers of t^n we get (41).

4. GENERAL SYMMETRY IDENTITIES FOR LAGUERRE POLY-BERNOULLI POLYNOMIALS

In this section, we give general symmetry identities for the Laguerre poly-Bernoulli polynomials ${}_LB_n^{(k)}(x, y, z)$ by applying the generating function (15). The result extend some known identities of Khan [7] and Pathan et al. [12]-[13].

Theorem 4.1. Let $a, b > 0$ and $a \neq b$. For $x, y, z \in R$ and $n \geq 0$. Then the following identity holds true:

$$\begin{aligned} & \sum_{m=0}^n \binom{n}{m} b^m a^{n-m} {}_LB_{n-m}^{(k)}(x, by, b^2z) {}_LB_m^{(k)}(x, ay, a^2z) \\ (42) \quad &= \sum_{m=0}^n \binom{n}{m} a^m b^{n-m} {}_LB_{n-m}^{(k)}(x, ay, a^2z) {}_LB_m^{(k)}(x, by, b^2z) \end{aligned}$$

Proof. Start with

$$(43) \quad g(t) = \left(\frac{(Li_k(1 - e^{-at}))(Li_k(1 - e^{-bt}))(C_0(xt))^2}{(e^{at} - 1)(e^{bt} - 1)} \right) e^{abyt + a^2b^2zt^2}$$

Then the expression for $g(t)$ is symmetric in a and b and we can expand $g(t)$ into series in two ways to obtain

$$\begin{aligned} g(t) &= \sum_{n=0}^{\infty} {}_LB_n^{(k)}(x, by, b^2z) \frac{(at)^n}{n!} \sum_{m=0}^{\infty} {}_LB_m^{(k)}(x, ay, a^2z) \frac{(bt)^m}{m!} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^n \binom{n}{m} a^{n-m} b^m {}_LB_m^{(k)}(x, by, b^2z) {}_LB_{n-m}^{(k)}(x, ay, a^2z) t^n \end{aligned}$$

On the similar lines we can show that

$$\begin{aligned} g(t) &= \sum_{n=0}^{\infty} {}_LB_n^{(k)}(x, ay, a^2z) \frac{(bt)^n}{n!} \sum_{m=0}^{\infty} {}_LB_m^{(k)}(x, by, b^2z) \frac{(at)^m}{m!} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^n \binom{n}{m} a^m b^{n-m} {}_LB_{n-m}^{(k)}(x, ay, a^2z) {}_LB_m^{(k)}(x, by, b^2z) t^n \end{aligned}$$

Comparing the coefficients of t^n on the right hand sides of the last two equations we arrive at the desired result.

Remark 1. By setting $b = 1$ in Theorem (4.1), we immediately following result

$$\sum_{m=0}^n \binom{n}{m} a^{n-m} {}_LB_{n-m}^{(k)}(x, y, z) {}_LB_m^{(k)}(x, ay, a^2z)$$

$$(44) \quad = \sum_{m=0}^n \binom{n}{m} a^m {}_L B_{n-m}^{(k)}(x, ay, a^2 z) {}_L B_m^{(k)}(x, y, z)$$

Theorem 4.2. Let $a, b > 0$ and $a \neq b$. For $x, y, z \in R$ and $n \geq 0$. Then the following identity holds true:

$$(45) \quad \sum_{m=0}^n \binom{n}{m} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} b^m a^{n-m} {}_L B_{n-m}^{(k)} \left(x, by + \frac{b}{a}i + j, b^2 u \right) {}_L B_m^{(k)}(x, az, a^2 v)$$

$$= \sum_{m=0}^n \binom{n}{m} \sum_{i=0}^{b-1} \sum_{j=0}^{a-1} a^m b^{n-m} {}_L B_{n-m}^{(k)} \left(x, ay + \frac{a}{b}i + j, a^2 u \right) {}_L B_m^{(k)}(x, bz, b^2 v)$$

Proof. Let

$$(46) \quad g(t) = \left(\frac{Li_k(1 - e^{-at}) Li_k(1 - e^{-bt}) (C_0(xt))^2}{(e^{at} - 1)(e^{bt} - 1)} \right) \left(\frac{(e^{abt} - 1)^2 e^{ab(y+z)t + a^2 b^2 (u+v)t^2}}{(e^{at} - 1)(e^{bt} - 1)} \right)$$

$$g(t) = \left(\frac{Li_k(1 - e^{-at}) C_0(xt)}{(e^{at} - 1)} \right) e^{abyt + a^2 b^2 ut^2} \left(\frac{e^{abt} - 1}{e^{bt} - 1} \right)$$

$$\times \left(\frac{Li_k(1 - e^{-bt}) C_0(xt)}{(e^{bt} - 1)} \right) e^{abzt + a^2 b^2 vt^2} \left(\frac{e^{abt} - 1}{e^{at} - 1} \right)$$

$$g(t) = \left(\frac{Li_k(1 - e^{-at}) C_0(xt)}{(e^{at} - 1)} \right) e^{abyt + a^2 b^2 ut^2} \sum_{i=0}^{a-1} e^{bti}$$

$$(47) \quad \times \left(\frac{Li_k(1 - e^{-bt}) C_0(xt)}{(e^{bt} - 1)} \right) e^{abzt + a^2 b^2 vt^2} \sum_{j=0}^{b-1} e^{atj}$$

$$= \left(\frac{Li_k(1 - e^{-at}) C_0(xt)}{(e^{at} - 1)} \right) e^{a^2 b^2 ut^2} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} e^{(by + \frac{b}{a}i + j)at} \sum_{m=0}^{\infty} {}_L B_m^{(k)}(x, az, a^2 v) \frac{(bt)^m}{m!}$$

$$= \sum_{n=0}^{\infty} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} {}_L B_{n-m}^{(k)} \left(x, by + \frac{b}{a}i + j, b^2 u \right) \frac{(at)^n}{n!} \sum_{m=0}^{\infty} {}_L B_m^{(k)}(x, az, a^2 v) \frac{(bt)^m}{m!}$$

$$(48) \quad = \sum_{n=0}^{\infty} \sum_{m=0}^n \binom{n}{m} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} {}_L B_{n-m}^{(k)} \left(x, by + \frac{b}{a}i + j, b^2 u \right) \sum_{m=0}^{\infty} {}_L B_m^{(k)}(x, az, a^2 v) b^m a^{n-m} t^n$$

On the other hand

$$(48) \quad g(t) = \sum_{n=0}^{\infty} \sum_{m=0}^n \binom{n}{m} \sum_{i=0}^{b-1} \sum_{j=0}^{a-1} {}_L B_{n-m}^{(k)} \left(x, ay + \frac{a}{b}i + j, a^2 u \right) \sum_{m=0}^{\infty} {}_L B_m^{(k)}(x, bz, b^2 v) b^{n-m} a^m t^n$$

By comparing the coefficients of t^n on the right hand sides of the last two equations, we arrive at the desired result.

Theorem 4.3. Let $a, b > 0$ and $a \neq b$. For $x, y, z \in R$ and $n \geq 0$. Then the following identity holds true:

$$\begin{aligned} & \sum_{m=0}^n \binom{n}{m} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} b^m a^{n-m} {}_L B_{n-m}^{(k)} \left(x, by + \frac{b}{a}i + j, b^2u \right) {}_L B_m^{(k)} \left(x, az + \frac{a}{b}j, a^2v \right) \\ (49) \quad & = \sum_{m=0}^n \binom{n}{m} \sum_{i=0}^{b-1} \sum_{j=0}^{a-1} a^m b^{n-m} {}_L B_{n-m}^{(k)} \left(x, ay + \frac{a}{b}i + j, a^2u \right) {}_L B_m^{(k)} \left(x, bz + \frac{b}{a}j, b^2v \right) \end{aligned}$$

Proof. The proof is analogous to Theorem (4.2) but we need to write equation (46) in the form

$$(50) \quad g(t) = \sum_{n=0}^{\infty} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} {}_L B_{n-m}^{(k)} \left(x, by + \frac{b}{a}i + j, b^2u \right) \frac{(at)^n}{n!} \sum_{m=0}^{\infty} {}_L B_m^{(k)} \left(x, az + \frac{a}{b}j, a^2v \right) \frac{(bt)^m}{m!}$$

On the other hand, equation (46) can be shown equal to

$$(51) \quad g(t) = \sum_{n=0}^{\infty} \sum_{i=0}^{b-1} \sum_{j=0}^{a-1} {}_L B_{n-m}^{(k)} \left(x, ay + \frac{a}{b}i + j, a^2u \right) \frac{(bt)^n}{n!} \sum_{m=0}^{\infty} {}_L B_m^{(k)} \left(x, bz + \frac{b}{a}j, b^2v \right) \frac{(at)^m}{m!}$$

Next making change of index and by equating the coefficients of t^n to zero in (50) and (51), we get the result.

REFERENCES

- [1] L.C. Andrews, *Special functions for engineers and mathematicians*, Macmillan. Co. New York, 1985.
- [2] T.M. Apostol, *On the Lerch zeta function*, Pacific J.Math. 1 (1951), 161-167.
- [3] E.T. Bell, *Exponential polynomials*, Ann. of Math. 35 (1934), 258-277.
- [4] G. Dattoli, S. Lorenzutta and C. Cesarano, *Finite sums and generalized forms of Bernoulli polynomials*, Rendiconti di Matematica, 19 (1999), 385-391.
- [5] G. Dattoli and A. Torre, *Operational methods and two variable Laguerre polynomials*, Atti Acadmia di Torino, 132 (1998), 1-7.
- [6] M. Kaneko, *Poly-Bernoulli numbers*, J.de Theorie de Nombres 9 (1997), 221-228.
- [7] W.A. Khan, *Some properties of the generalized Apostol type Hermite-based polynomials*, Kyungpook Math. J. 55 (2015), 597-614.
- [8] T. Kim, *Some identities for the Bernoulli, the Euler and the Genocchi numbers and polynomials*, Adv. Stud. Contemp. Math. 20 (1) (2010), 23-28.
- [9] T. Kim, Y.S. Jang and J.J. Seo, *poly-Bernoulli polynomials and their applications*, Int. Journal of Math. Analysis, 8 (2014), No. 30, 1495-1503.
- [10] T. Kim, H.K. Kwon, S.H. Lee and J.J. Seo, *A note on poly-Bernoulli numbers and polynomials of the second kind*, Adv. Difference Equ. (2014), 214-219.
- [11] D.S. Kim and T. Kim, *A note on poly-Bernoulli and higher order poly-Bernoulli polynomials*, Russian J. Math. Phy. 22 (1) (2015), 26-33.

- [12] M.A. Pathan and W.A. Khan, *A new class of generalized polynomials associated with Hermite and Bernoulli polynomials*, LE MATEMATICHE, Vol. LXX (2015), 53-70.
- [13] M.A. Pathan and W.A. Khan, *A new class of generalized Hermite-Bernoulli polynomials*, Georgian Math. J. 19 (2012), 559-573.
- [14] E.D. Rainville, *Special functions*, The MacMillan Comp., New York, (1960).
- [15] H.M. Srivastava and H.L. Manocha, *A treatise on generating functions*, Ellis Horwood Limited, New York, 1984.

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